THE RELATIVISTIC GARDENHOSE INSTABILITY II*

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Abstract

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For an anisotropic, spatially uniform, ultra-relativistic plasma, composed of protons and anti-protons (or electrons and positrons), embedded in an infinite, homogeneous magnetic field, we investigate the assumptions that are usually made in discussing the gardenhose instability in the long wavelength, low frequency regime.

It is shown that for a particular non-Maxwellian distribution function the usual criterion for instability does not result. Instead of obtaining

$$4\pi\left(P_{\parallel}-P_{\perp}\right)>H_0^2$$

we obtain $P_{\perp} > P_{\parallel}$ for instability.

Thus for non-Maxwellian plasmas we feel that some care should be exercized in dealing with the gardenhose mode.

Author

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I. Introduction

In a recent paper (Lerche, 1966), hereafter referred to as I, we investigated the conditions required of a relativistic plasma, embedded in an infinite, homogeneous magnetic field, in order that the 'gardenhose' mode give rise to an unstable situation. It was shown that in the long wavelength, low frequency limit Parker's (1958) instability condition $4\pi(P_{\parallel} - P_{\perp}) > H_0^2$ * was recovered, even when a relativistic plasma component was present.

In deriving the results presented in I the usual approach was followed. Namely a small- k and small- ω expansion of the appropriate resonance denominators was performed inside the various integrals which arose. It was then tacitly assumed that the resulting integrals converged. Retaining only the lowers powers of k and ω the usual instability condition was derived. In this paper we propose to investigate the validity of this assumption. We will show that in at least one particular case the usual condition for instability of the gardenhose mode is suspect and thus the usual approach of a small- k and small- ω expansion should be treated with some caution.

The point is that a small- k expansion is valid provided the first few coefficients of powers of k become progressively smaller since one is then performing an asymptotic expansion. However we will demonstrate that for at least one particular plasma such an expansion breaks down since only the first power of k has a finite coefficient. In fact we shall show that the usual condition for instability of the gardenhose mode is not applicable in this one particular case. This raises doubt concerning its validity for any plasma. We contend that if the usual condition is invalid for one particular situation then it should be treated with

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* The notation employed throughout this paper is the same as in I.

some caution in all other situations.

In particular we shall concern ourselves only with the case of an ultra-relativistic proton (or electron) plasma which is taken to be spatially uniform but anisotropic. We assume that there co-exists with this plasma an anti-proton (or positron) plasma which is also ultra-relativistic and possesses exactly the same distribution function as the proton (electron) plasma. Both species of particles are taken to have zero streaming velocity in any direction. The composite plasma is assumed to be embedded in an ambient magnetic field, which is homogeneous and infinite in extent. For definiteness we treat the case of a proton and anti-proton plasma.

As a particular application of this calculation we have in mind the galactic cosmic ray plasma. However no attempt will be made to consider any particular plasma in the present paper.

We realize that the analysis is restrictive since the assumption is made throughout that the two species of particles have identical distribution functions. However the analysis will serve to demonstrate that the marginally unstable waves do not yield the usual gardenhose conditions for instability in either the long wavelength (Parker, 1958) or finite wavelength (Noerdlinger, 1963, 1966; Sudan 1963, 1965) limits. We will argue that the long wavelength limit is not approached uniformly for any equilibrium distribution function. Thus the choice of distribution function affects significantly the curve of marginal stability. As a consequence the analysis given here of the gardenhose mode is by no means complete. It does have the advantage that it demonstrates, in no uncertain manner, the strong dependence of the results on the choice of distribution function.

2. The Dispersion Relation

It is a simple matter to show that the dispersion relation for the relativistic gardenhose mode, when two mobile species, with identical distribution functions but differing in the sign of their charge, are present, can be written (Montgomery and Tidman, 1964)

$$c^2k^2 - \omega^2 - \frac{1}{2}\pi\, \omega_p^2 \times$$
$$\int_0^\infty dp_\perp \int_{-\infty}^\infty dp_\parallel \frac{p_\perp^2 \left[\left(\omega\sqrt{(1+p^2)} - ckp_\parallel\right) \partial f_0/\partial p_\perp + ckp_\perp\, \partial f_0/\partial p_\parallel\right]}{\sqrt{(1+p^2)}\left[\omega\sqrt{(1+p^2)} - ckp_\parallel \pm \omega_L\right]} = 0 . \tag{1}$$

As in I the presence of $(\pm\omega_L)$ in the denominator indicates that the double integral is to be regarded as the sum of two terms containing $(+\omega_L)$ and $(-\omega_L)$ respectively.

In a general plasma where the mobile species have different rest masses and different distribution function the dispersion relation is much more complicated than (1) (Noerdlinger, 1965).

The reason for choosing the particular situation being considered is primarily due to the simplification of the general dispersion relation in such a case.

The equilibrium distribution function, f_0, has been normalized so that

$$2\pi\int_0^\infty dp_\perp \int_{-\infty}^\infty dp_\parallel\, p_\perp f_0 = 1 , \tag{2}$$

and f_0 is taken to be independent of the phase angle φ, as in I.

It now proves more convenient to transform to spherical momentum coordinates in (1). These are defined by

$$p_{\parallel} = p\cos\psi, \quad p_{\perp} = p\sin\psi .$$

It can then be shown that (1) may be written

$$c^2k^2 - \omega^2 - \frac{1}{2}\pi\omega_p^2 \times$$

$$\int_{-1}^{+1} d\mu \int_0^{\infty} dp \frac{(1-\mu^2)p^3\left[\omega\sqrt{(1+p^2)}\,\partial f_0/\partial p + (ck - \omega\mu p^{-1}\sqrt{(1+p^2)})\,\partial f_0/\partial\mu\right]}{\sqrt{(1+p^2)}\left[\omega\sqrt{(1+p^2)} - ckp\mu \pm \omega_L\right]} = 0, \tag{3}$$

where $\mu = \cos\psi$.

While it would be useful if (3) could be analyzed for all momentum values once f_0 is given, this is a rather difficult task due to the presence of the branch cut which appears in the p integrand. Some physical insight can be gained in three extreme cases.

(i) If we deal with a thermal plasma only we can replace $\sqrt{(1+p^2)}$ by unity in the double integrand. This case has been discussed in detail elsewhere (Rosenbluth, 1956; Parker, 1958; Vedenov and Sagdeev, 1958; Noerdlinger, 1963).

(ii) If the plasma is taken to be only weakly relativistic we can replace $\sqrt{(1+p^2)}$ by $1 + \frac{1}{2}p^2$ in the double integrand. This case, and variations of it e.g. $\sqrt{(1+p^2)} \simeq \sqrt{(1+p_{\perp}^2)} + \frac{1}{2}p_{\parallel}^2$, have also received considerable attention (Noerdlinger, 1963, 1966; Sudan 1963, 1965).

(iii) If the plasma is ultra-relativistic we can replace $\sqrt{(1+p^2)}$ by p throughout the analysis. This case emphasizes as much as possible the relativistic nature of the plasma and will form the basis of this paper.

Under approximation (iii) we see that (3) becomes

$$c^2k^2-\omega^2-\frac{1}{2}\pi\omega_p^2 \times \int_{-1}^{+1} d\mu \int_0^\infty dp \, \frac{(1-\mu^2)\,p^2\left[\omega p\,\partial f_0/\partial p+(ck-\omega\mu)\,\partial f_0/\partial\mu\right]}{(\omega p - ckp\mu \pm \omega_L)} = 0 . \quad (4)$$

Before further progress can be made with (4) we must specify the equilibrium distribution function, f_0 . The conventional approach at this juncture is to assume that f_0 is some form of two-temperature Maxwellian distribution. Such a distribution function unfortunately leads to integrals which have to be evaluated approximately, either analytically or numerically.

Let us remember that the main application of any result is directed towards an understanding of the galactic cosmic ray gas. It is known from observation that the galactic cosmic ray gas does not possess a Maxwellian distribution function. In fact the cosmic ray gas distribution function behaves as an inverse power law at high momentum values. Consequently in order to emulate the behavior of the cosmic ray gas in some respects we let

$$f_0 = \zeta\, p(1+\alpha\mu^2)(p^2+p_0^2)^{-3}, \quad (5)$$

for both protons and anti-protons, where ζ , α and p_0 are constants.

This particular distribution function has the advantage that it enables the various integrals occurring in (5) to be performed analytically. At the same time it gives an idea of the plasma behavior due to the inclusion of some degree of anisotropy through the factor $(1+\alpha\mu^2)$. We emphasize that (5) is not intended to represent the observed cosmic ray spectrum at any momentum value. It does possess some of the gross features of the observed spectrum in that it turns into an inverse power law at high momentum values and it also possesses a maximum in the range $0<p<\infty$ as does the observed cosmic ray spectrum (Parker, 1965).

It can be argued that the p range of integration in (4) should be terminated at some lower momentum value since we are using the ultra-relativistic approximation. However we will see that when a small- k expansion is taken, the resulting divergence of various integrals results from the high momentum values. Thus the use of the ultra-relativistic approximation does not cause the non-uniform results.

Use of (5) in (4) enables us to write

$$c^2k^2-\omega^2-\frac{1}{2}\pi\omega_p^2 S\int_0^\infty dp\int_{-1}^{+1}d\mu\; p^3(1-\mu^2)\left[\omega(p_0^2-5p^2)(1+\alpha\mu^2)+2\alpha\mu(p_0^2+p^2)(ck-\omega\mu)\right]\times$$

$$(p_0^2+p^2)^{-4}\left[(\omega p-ckp\mu+\omega_L)^{-1}+(\omega p-ckp\mu-\omega_L)^{-1}\right] = 0\,. \qquad (6)$$

Without loss of generality we define k to be real and positive and we initially define ω to lie in the upper half complex plane. It can then be shown (Penrose, 1960) that a necessary and sufficient condition for an unstable situation to develop is that the imaginary part of (6) vanish for some real $\omega \geqslant 0$, provided only that the real part of (6) yields a real, positive k

for this ω .

Since ω is initially defined to lie in the upper half complex plane we see that as $\mathrm{Im}(\omega) \longrightarrow 0$ from above we must understand

$$X^{-1} \equiv P(X^{-1}) + i\pi\delta(X) \tag{7}$$

where $X = p\omega - ckp\mu \pm \omega_L$

Here $P(X^{-1})$ denotes the principal value of X^{-1} and $\delta(X)$ is the Dirac δ -function.

Regarding the denominators in the integrands of (6) as functions of μ we see that the first double integral possesses a pole in the μ range of integration, for ω real, at

$$\mu = (p\omega + \omega_L)/(ckp) \equiv \mu_1, \text{ say} \tag{8}$$

provided $\omega < ck$ and $p \geqslant \omega_L (ck-\omega)^{-1} \equiv p_1$, say.

Likewise the second double integral possesses a pole at

$$\mu = (p\omega - \omega_L)/(ckp) \equiv \mu_2, \text{ say} \tag{9}$$

provided $\omega < ck$ and $p \geqslant \omega_L (ck+\omega)^{-1} \equiv p_2$, say.

Setting

$$I_1(\mu_0) = P\int_{-1}^{+1} \frac{(1-\mu^2)\,d\mu}{(\mu-\mu_0)} \equiv -2\mu_0 + (1-\mu_0^2)\ln\left|\frac{1-\mu_0}{1+\mu_0}\right| , \tag{10a}$$

$$I_2(\mu_0) = P\int_{-1}^{+1} \frac{\mu(1-\mu^2)\,d\mu}{(\mu-\mu_0)} \equiv 4/3 + \mu_0 I_1(\mu_0), \tag{10b}$$

$$I_2(\mu_0) = P\int_{-1}^{+1} \frac{\mu^2(1-\mu^2)\,d\mu}{(\mu-\mu_0)} \equiv \mu_0 I_2(\mu_0) , \tag{10c}$$

we see that, for ω real, the real part of (6) is given by

$$c^2k^2 - \omega^2 + \frac{1}{2}\pi \zeta \omega_p^2 (ck)^{-1} \int_0^\infty dp \; p^2(p^2+p_0^2)^{-4} \Big\{ \omega(p_0^2-5p^2)[I_1(\mu_1)+I_1(\mu_2)] +$$

$$2\alpha ck(p_0^2+p^2)\big[I_2(\mu_1)+I_2(\mu_2)\big] - \alpha\omega(p_0^2+7p^2)\big[I_3(\mu_1)+I_3(\mu_2)\big]\Big\} = 0. \tag{11}$$

The imaginary part of (6), for real ω, is given by

$$\frac{1}{2}\pi^2 \zeta \omega_p^2 (ck)^{-1} \Big\{ \int_{p_1}^\infty dp \; p^2(1-\mu_1^2)(p^2+p_0^2)^{-4}\big[\omega(p_0^2-5p^2) + 2\alpha ck\mu_1(p_0^2+p^2) - \alpha\omega\mu_1^2(p_0^2+7p^2)\big]$$

$$+ \int_{p_2}^\infty dp \; p^2(1-\mu_2^2)(p^2+p_0^2)^{-4}\big[\omega(p_0^2-5p^2) + 2\alpha\mu_2 ck(p_0^2+p^2) - \alpha\omega\mu_2^2(p_1^2+7p^2)\big]\Big\} = 0. \tag{12}$$

It can be seen by inspection that (12) vanishes when $\omega = 0$. Thus marginally unstable waves are associated with zero phase velocity. The vanishing of (12) for some real $\omega \geqslant 0$ is a necessary condition for instability but it is not sufficient. We must also demand that such a marginally unstable wave give rise to a positive, definite wave number. In fact the region of unstable waves can easily be shown from (11) to satisfy

$$c^2k^2 + 2\pi \zeta\alpha \omega_p^2 \int_0^\infty dp \; p^2 I_2\left(\omega_L/(ckp)\right)(p^2+p_0^2)^{-3} \leq 0, \text{ unstable.} \tag{13}$$

Upon making use of the analytic form for I_2 we see that (13) becomes

$$c^2k^2 + 2\pi\zeta\alpha\omega_p^2 \times$$

$$\int_0^\infty dp \; p^2(p^2+p_0^2)^{-3}\left[4/3 - \frac{2\omega_L^2}{(ckp)^2} + \frac{\omega_L}{(ckp)}\left(1 - \frac{\omega_L^2}{c^2k^2p^2}\right)\ln\left|\frac{ckp-\omega_L}{ckp+\omega_L}\right|\right] \leqslant 0, \text{ unstable.} \tag{14}$$

It is at this point that the singular nature of the long wavelength limit becomes apparent. Suppose we assume that k is extremely small and expand <u>inside</u> the integral retaining only the lowest power of k which occurs. Then it is a simple matter to show that, to order k^2, (14) becomes

$$c^2k^2\left[1 - 4\pi H_0^{-2}\left(P_{\parallel} - P_{\perp}\right)\right] \leqslant 0, \text{ unstable,} \tag{15}$$

where the parallel and perpendicular pressure components due to both the protons and anti-protons are given respectively by

$$P_{\parallel} = 4\pi N mc^2 \zeta \int_{-1}^{+1} d\mu \int_0^{\infty} dp \, \frac{p^5 \mu^2 (1+\alpha\mu^2)}{\sqrt{(1+p^2)}\,(p^2+p_0^2)^3} , \tag{16a}$$

$$P_{\perp} = 2\pi N mc^2 \zeta \int_{-1}^{+1} d\mu \int_0^{\infty} dp \, \frac{p^5 (1-\mu^2)(1+\alpha\mu^2)}{\sqrt{(1+p^2)}\,(p^2+p_0^2)^3} . \tag{16b}$$

Thus, to order k^2, it would appear that the usual long wavelength condition for instability is recovered. However we have assumed implicitly that all remaining powers of k in the small-k expansion have finite coefficients. <u>For the particular distribution function chosen this is not true.</u> In fact the coefficient of k^4 is infinite.

Thus the validity of expanding the integrand in powers of k and performing term by term integration is suspect. Only for the particular class of distribution function which decay exponentially at high momentum values has the author been able to show that all coefficients of all powers of k remain finite in such a small-k expansion. (Even then the coefficients increase without limit as one includes all k terms. This increase can be tolerated if the first few terms converge and become progressively smaller since one is then performing an asymptotic expansion.)

The limit of small-k certainly does <u>not</u> yield the usual result for the distribution function given by (5) as we will now show. If we let

$\beta = \omega_L (ck)^{-1}$, $y = p_0 \beta^{-1}$, $p = \beta x$ it can be seen that (14) becomes

$$c^2k^2 + 2\pi \Im \alpha \omega_p^2 \beta^{-3} \int_0^\infty x^2 (x^2+y^2)^{-3} \left[4/3 - 2x^{-2} + x^{-3}(x^2-1) \ln \left| \frac{x-1}{x+1} \right| \right] dx$$

$$\lessgtr 0, \text{ unstable.} \tag{17}$$

It can easily be shown that

$$\int_0^\infty x^2 (x^2+y^2)^{-3} dx = \pi/(16y^3) \quad , \tag{18a}$$

and

$$\int_0^\infty (x^2+y^2)^{-3} dx = 3\pi/(16y^5) \quad . \tag{18b}$$

The remaining integral in (17) is rather more difficult to evaluate. It is shown in the appendix that

$$\int_0^\infty x^{-1}(x^2-1)(x^2+y^2)^{-3} \ln \left| \frac{1-x}{1+x} \right| dx = \pi^2/(2y^6) \tag{18c}$$

Making use of (18) in (17) we see that for instability we require

$$c^2k^2 + \frac{\pi^2 \alpha \Im \omega_p^2}{6 p_0^3} \left(1 - \tfrac{9}{2} \beta^2 p_0^{-2} + 6\pi \beta^3 p_0^{-3} \right) \lessgtr 0, \text{ unstable.} \tag{19}$$

With $\beta = q\, p_0$ this relation becomes

$$q^{-2} + \frac{\pi^2 \zeta \alpha\, \omega_p^2}{6\, p_0\, \omega_L^2}\left(1 - \tfrac{9}{2} q^2 + 6\pi q^3\right) \leq 0, \text{ unstable.} \tag{20}$$

Upon evaluation of (16) in the ultra-relativistic limit we see that

$$P_\parallel = \pi^2 \zeta N m c^2 p_0^{-1}\left(1 + 3\alpha/5\right), \tag{21a}$$

$$P_\perp = \pi^2 \zeta N m c^2 p_0^{-1}\left(1 + \alpha/5\right). \tag{21b}$$

Thus (20) reduces to

$$q^{-2} + \frac{5}{3}\left(P_\parallel - P_\perp\right) 4\pi H_0^{-2}\left(1 - \tfrac{9}{2} q^2 + 6\pi q^3\right) \leq 0, \text{ unstable.} \tag{22}$$

We note that

$$1 - \tfrac{9}{2} q^2 + 6\pi q^3$$

is a positive definite function of q for $q \geq 0$ i.e. real, positive wave numbers.

Thus an unstable situation will develop for this distribution function if, and only if, $P_\perp > P_\parallel$. This result is in contrast to the usual result obtained for the gardenhose mode which demands

$$4\pi\left(P_\parallel - P_\perp\right) > H_0^2, \tag{23}$$

for an unstable situation.

If we assume $P_\perp > P_\parallel$ we see that the unstable waves satisfy

$$q^{-2} - R\left(1 - \tfrac{9}{2} q^2 + 6\pi q^3\right) \leq 0, \text{ unstable} \tag{24}$$

where

$$R = \frac{5}{3}\left(P_\perp - P_\parallel\right) 4\pi H_0^{-2} > 0 \quad . \tag{25}$$

At small wavelengths it is clear that the left hand side of (24) is positive while at large wavelengths it is negative. Thus only wavelengths greater than some minimum are unstable.

Since (25) is a quintic equation in q for the marginally unstable waves it is analytically impossible to find the minimum unstable wavelength exactly. However some progress can be made when R is much less than, or much greater than, unity.

(a) $R >> 1$

In this case those wavelengths are unstable for which

$$k \lesssim \omega_L R^{1/2} (c p_0)^{-1} = \frac{|\epsilon|}{mc^2 p_0} \sqrt{\left(\frac{10\pi}{3}\left(P_\perp - P_\parallel\right)\right)} \equiv k_0, \text{ say} . \tag{26}$$

Note that in this limit the criterion for instability is approximately independent of the ambient magnetic field.

(b) $R \ll 1$

Here instability sets in for waves such that

$$k \lesssim \omega_L (cp_0)^{-1} \left(6\pi R\right)^{1/5} = \frac{2\pi|\epsilon| H_0}{mc^2 p_0}\left[5 H_0^{-2}(P_\perp - P_\parallel)\right]^{1/5} \equiv k_1 \text{, say.} \tag{27}$$

These conditions for instability differ markedly from those obtained by previous investigators who, as far as the author is aware, restricted their attention to Maxwellian type distribution functions. It is interesting to note that if the plasma is isotropic $(\alpha = 0)$ then although the imaginary part of (6) vanishes for $\omega = 0$ there is no corresponding real k. As a consequence an isotropic plasma is stable against the gardenhose mode.

3. Conclusion

We have investigated the validity of the assumptions which are usually made in discussing the gardenhose instability. It has been shown that for a particular plasma distribution function, which is non-Maxwellian, the assumptions are invalid. In particular the condition for instability is completely altered so that an unstable situation will develop if, and only if, $P_\perp > P_\parallel$ for this mode.

In view of this result the author feels that the presence, or absence, of the usual gardenhose instability is strongly dependent on the choice of distribution function. In particular he feels that for non-Maxwellian plasmas some care should be exercised when dealing with the gardenhose mode.

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Appendix

Let

$$I = \int_0^\infty x^{-1}(x^2-1)(x^2+y^2)^{-3} \ln\left|\frac{1-x}{1+x}\right| dx \quad . \tag{A1}$$

Then

$$I = \int_0^1 \frac{x^{-1}(x^2-1)\ln(1-x)dx}{(x^2+y^2)^3} + \int_1^\infty \frac{x^{-1}(x^2-1)\ln(x-1)dx}{(x^2+y^2)^3} - \int_0^\infty \frac{\ln(1+x)\, x^{-1}(x^2-1)}{(x^2+y^2)^3}dx \, . \tag{A2}$$

In the first integral of (A2) we substitute $z = 1-x$, in the second we use $z = x-1$ and in the third we put $z = 1+x$. Then

$$I = \int_0^1 \frac{z(z-2)\ln z\, dz}{(1-z)\left[(z-1)^2+y^2\right]^3} + \int_0^\infty \frac{z(z+2)\ln z\, dz}{(z+1)\left[(z+1)^2+y^2\right]^3} - \int_1^\infty \frac{z(z-2)\ln z\, dz}{(z-1)\left[(z-1)^2+y^2\right]^3} \, . \tag{A3}$$

We write

$$\int_1^\infty \frac{z(z-2)\ln z\, dz}{(z-1)\left[(z-1)^2+y^2\right]^3} = \int_0^\infty \frac{z(z-2)\ln z\, dz}{(z-1)\left[(z-1)^2+y^2\right]^3} + \int_0^1 \frac{z(z-2)\ln z\, dz}{(1-z)\left[(z-1)^2+y^2\right]^3} \, . \tag{A4}$$

Use of (A4) in (A3) leads to

$$I = \int_0^\infty \left\{ \frac{(z+2)}{(z+1)\left[(z+1)^2+y^2\right]^3} - \frac{(z-2)}{(z-1)\left[(z-1)^2+y^2\right]^3} \right\} z \ln z\, dz \, , \tag{A5}$$

or

$$I = \frac{1}{2}\frac{\partial^2}{\partial(y^2)^2}\int_0^\infty \left\{\frac{(z+2)}{(z+1)\left[(z+1)^2+y^2\right]} - \frac{(z-2)}{(z-1)\left[(z-1)^2+y^2\right]}\right\} z \ln z \, dz. \tag{A6}$$

This integral can be performed with no difficulty and upon evaluation we find

$$I = \frac{1}{2}\frac{\partial^2}{\partial(y^2)^2}\left[\frac{\pi^2}{2y^2}(1-y^2)\right] . \tag{A7}$$

Thus

$$I = \pi^2/(2y^6) , \tag{A8}$$

which is the required result.